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CLOSURE OF THE EQUATIONS OF TURBULENCE USING THE ANALYTIC AND SCALING PROPERTIES OF THE SPECTRAL FUNCTIONS

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We discuss a spectral method of closing the equations of fully-developed shear turbulence based on the hypotheses of scaling invariance of the long-wavelength fluctuations of the velocity field and factorization of the dependence of the spectral functions on the magnitude and orientation of the wave vector k. It is also assumed that certain universal scaling functions appearing in the parametrization are analytic in k, rather than the individual components of the spectral tensors. It is shown that with these assumptions the turbulent structure is described locally by a small number of secular parameters, for which a relatively simple system of quasilinear differential equations is derived. In addition to the correlation length, mean rate of energy dissipation, and the Reynolds stress tensor (as in the semi-empirical models) the secular quantities also include the "fast" part of the pressure–deformation-rate correlation tensor, or equivalently the second orientation moments of the spectral function F_{ij}.

It was shown in [1] that the structure of fully developed isotropic turbulence produced by a grid in the long-wavelength region can be described by two secular fields: the mean rate of energy dissipation $\langle\varepsilon\rangle$ and the correlation length r_c.

The possibility of a simple description of this kind is intimately connected with the assumption that turbulent flow behaves as a critical system. The basis for this analogy is the similarity of the large-scale disturbances of the velocity field and the existence of a power-law (with exponent β) part of the spectrum in the inertial region. The intrinsic scales of length and time are the dissipative (Kolmogorov) quantities $r_d = (\eta^3/\langle\varepsilon\rangle)^{1/4}$ and $t_d = (\eta/\langle\varepsilon\rangle)^{1/2}$. The scaling dimensionality a of the velocity field in the approximation $\beta = 5/3$ is equal to 1/3, while the scaling dimensionality of the field ε in the same approximation is $(-\mu/2)$. Here μ is the spectral index characterizing energy dissipation fluctuations and η is the kinematic viscosity.

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In this analogy one assumes that the spectral functions have the scaling form in the limit $\bar{k} \ll 1$

$$\bar{F} = \bar{r}_c^{2a}\varphi(kr_c), \quad \bar{T} = \bar{r}_c^{3a}t(kr_c). \tag{1}$$

Here F = F(x, k) and T = T(x, k) characterize two-point quadratic and cubic velocity correlations; x is the distance behind the grid; $k = |\mathbf{k}|$; $\mathbf{k}$ is the wave vector; the bar means that the quantities have been made dimensionless with the help of the length and time scales r_d and t_d. The tensors $F_{ij}(\mathbf{x}, \mathbf{k}) = (2\pi)^{-3}\int \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r})\rangle \exp(-i\mathbf{kr})\, d\mathbf{r}$ and $T_{i\ell,j}(\mathbf{x}, \mathbf{k})$ (defined analogously) are related to F and T by

$$F_{ij} = p_{ij}F, \; T_{il,j} = i(p_{ij}\theta_l + p_{lj}\theta_i)T, \tag{2}$$

where $p_{ij} = \delta_{ij} - \theta_i\theta_j$; $\theta_i = k_i/k$. It follows from (1) and (2) that the spectral functions depend on x implicitly through $\langle\varepsilon\rangle$ and r_c, which means that (in the terminology of statistical physics) the fields $\langle\varepsilon\rangle$ and r_c are secular.

The dependence of $\langle\varepsilon\rangle$ and r_c on x can be determined using data on the small-k behavior of the universal functions ϕ and t. Following [2], we assume the analyticity condition for these functions and

$$\varphi \to \text{const} \neq 0, \; t \to 0 \text{ for } kr_c \to 0. \tag{3}$$

The constant in the first relation of (3) is taken to be unity. We note that according to (3) the functions F_{ij}, $T_{i\ell,j}$,... are nonanalytic when $kr_c \to 0$; they have dipole-type singularities. This is consistent with the existence of hydrodynamic long-range action in an incompressible fluid.

A direct analysis of the spectral equations with the application of (3) leads to a power-law dependence of the basic turbulent characteristics on x [3]. In particular, the values $48/(40 - 3\mu)$ and $-16/(40 - 3\mu)$ are obtained for the damping exponents of the quantities $\langle u_i^2\rangle$ and r_c. These values are very close to the experimental values and become -1.2 and 0.4 in the Kolmogorov approximation when $\mu = 0$. It was also shown in [3] that when (3) is taken into account only the single parameter $\tau = (t_d/2)\, U\, (d \ln r_c/dx) \sim Re^{-1/2}$ remains in the spectral equations (U is the average velocity and Re is the Reynolds number). The quantity τ is the ratio of t_d to the "external" time scale. In the analogy with critical phenomena it is natural to identify τ with the reduced temperature measuring the closeness of the system to the critical point (which in our case corresponds to the limit Re $\to \infty$) and to use the usual representation

$$\bar{r}_c = \tau^{-\nu} \tag{4}$$

($\nu = 6/(4 + 3\mu)$ is the critical index determining the scaling dimensionality of the field τ [1]).

To extend this method to shear flow it is first necessary to deal with the anisotropic form of the spectral tensors, i.e., to generalize the representation (2). The known solutions of this problem are based on direct parametrization: besides θ, additional tensor arguments of the spectral functions are assumed, such as the Reynolds stress tensor $\langle u_i u_j\rangle$ ($\mathbf{u}$ is the velocity fluctuation) [4-9] or the tensor $f_{ij}^{(0)}$ obtained by integrating F_{ij} over all possible orientations of the vector θ [10].

Important conclusions about the θ-dependence of the spectral functions can be made without having to assume particular forms of the dependence. Indeed, a great deal of experimental data suggests a lack of isotropy in the inertial region [11, 12]. But in this region the longitudinal and transverse spectra, which differ essentially in the orientation structure, obey the "five-thirds law". This can only be possible if the dependence of F_{ij} and the higher-order tensors on kr_c can be factorized from the dependence on the other arguments:

$$F_{ij}(\mathbf{x}, \mathbf{k}) = f_{ij}(\mathbf{k}, \theta)\varphi(kr_c); \tag{5}$$
$$T_{il,j}(\mathbf{x}, \mathbf{k}) = f_{il,j}(\mathbf{x}, \theta)t(kr_c). \tag{6}$$

The hypothesis (5), (6) is indirectly supported by the dependence of the Kologorov constant on the anisotropy parameters [13]; however a direct test using available experimental data is a difficult problem. On the other hand, the functions f_{ij} and $f_{i\ell,j}$, which characterize the form of the turbulent structures, can easily be related to directly measureable quantities. For example, integrating (5) over all θ and using the fact that the long-wavelength region makes the dominant contribution to the integral, it is not difficult to obtain

$$\langle u_i u_j \rangle = \alpha r_c^{-3} f_{ij}^{(0)} \tag{7}$$

($\alpha = \int_0^\infty \varphi(t)\, t^2\, dt$ is a universal constant determined by the form of the spectrum in the long-wavelength region).

We next discuss the set of secular quantities for shear turbulence. Here, as in the case of isotropic turbulence, we will base our treatment on the equations for the spectral functions, the first of which (for F_{ij}) can be written in the form [10, 14, 15]

$$U_k \frac{\partial F_{ij}}{\partial x_k} + (U_{il}F_{lj})_s - U_{lm}k_l \frac{\partial F_{ij}}{\partial k_m} - ik_l\,(T_{il,j})_s + 2\eta k^2 F_{ij} = = 2U_{lm}\,(\theta_i\theta_l F_{mj})_s - ik_l\theta_m\,(\theta_i T_{lm,j})_s. \tag{8}$$

Here we have taken into account the condition of local homogeneity $r_c \ll L$ (L is the external length of the problem) and $U_{ij} = \partial U_i/\partial x_j$; the index s stands for a symmetrization; summation is understood over repeating indices. The third and fourth terms on the left-hand side of (8) describe energy transfer over the spectrum [16], and the terms on the right hand side describe intercomponent transfer. We consider the steady case and assume that $\sqrt{\langle u_i^2 \rangle} \ll U$, which implies that the diffusion terms in the equations can be neglected and the average velocity field can be assumed as given and not distorted by the presence of turbulence. This condition is well satisfied in certain cases for flow distorted by a grid. Plane vortex-free distortion [10, 17, 18], contraction [19, 20], and uniform shear [21-23] have been studied extensively experimentally. In spite of their relative simplicity, these flow types are still of great interest theoretically, since they are prime examples of the interaction of mean shear flow with fluctuations. In addition, the experimental data obtained for these flow types are often used to choose numerical values of the constants in the semi-empirical models.

Integration of (8) over all k leads to a transport equation for the Reynolds tensor $\langle u_i u_j \rangle$, which involves the unknown term $\Phi_{ij} = \langle p\,\partial u_i/\partial x_j \rangle$ describing correlations between pressure and deformation-rate fluctuations. This term causes difficulties in modeling. It is usually represented as a sum of two parts, the first of which $\Phi_{ij,1}$ results from the nonlinear interaction between turbulent fluctuations, and the second $\Phi_{ij,2}$ involves the average deformation rate. The Rott approximation is often used for $\Phi_{ij,1}$ and the so-called quasi-isotropic model is often used for $\Phi_{ij,2}$. In the simplest variant of this model [24-26]

$$\Phi_{ij,2} = \mathrm{const}\,(D_{ij} - D\delta_{ij}/3) \tag{9}$$

($D_{ij} = -(\langle u_i u_k \rangle \partial U_j/\partial x_k)_s$ is a tensor describing the production of stress by the average shear and $D = D_{ii}$).

Returning to the basic problem of determining the set of secular quantities for shear turbulence and a system of equations for these quantities, we note that in addition to $\langle \varepsilon \rangle$ and r_c this set should include quantities characterizing the anisotropy, such as the orientation moments of the f functions. For example, for f_{ij}

$$f_{ij}^{(l\ldots m)}(\mathbf{x}) = \int f_{ij}(\mathbf{x}, \theta)\, \theta_l \ldots \theta_m\, d\theta. \tag{10}$$

The order of a moment is determined by the number of superscripts.

The second-order moments are directly related to the tensor $\Phi_{ij,2}$: multiplying (5) by $U_{\ell i}\theta_\ell\theta_m$ and integrating over all k, we obtain

$$\Phi_{ij,2} = 2U_{lm} f_{mj}^{(li)} \alpha r_c^{-3}. \tag{11}$$

The results of [27-29] show indirectly that only a small number of the lowest-order moments can appear in the set of secular quantities. In this connection we first consider the Cray equation [10], which is obtained from (3) by integrating over all possible orientations of θ:

$$U_k \frac{\partial f_{ij}^{(0)}}{\partial x_k}\varphi + U_k \frac{\partial \ln r_c}{\partial x_k} f_{ij}^{(0)} k \frac{d\varphi}{dk} + \left(U_{il} f_{lj}^{(0)}\right)_s \varphi - \varphi U_{lm} \int k_l \frac{\partial f_{ij}}{\partial k_m}\, d\theta - U_{lm} f_{ij}^{(lm)} k \frac{d\varphi}{dk} + 2\eta k^2 f_{ij}^{(0)} \varphi + \left(f_{il,j}^{(l)} - f_{lm,j}^{(lmi)}\right)_s kt = 2U_{lm}\left(f_{mj}^{(li)}\right)_s \varphi. \tag{12}$$

Besides $f_{ij}^{(0)}$ this equation involves higher-order moments of f_{ij}, as well as moments of the cubic correlations. However, the analysis of this equation can be simplified by taking into account the analytic properties of the scalar spectral functions ϕ and t [30-32]. Following the approach used above for isotropic turbulence, we first consider (12) in the limit $kr_c \to 0$. Using the condition (3), we find an equation for the zero-order moments

$$U_k \frac{\partial f_{ij}^{(0)}}{\partial x_k} + (U_{il} f_{lj}^{(0)})_s + K_{ij} = 2U_{lm} (f_{mj}^{(li)})_s \tag{13}$$

$$\left(K_{ij} = -U_{lm} \int k_l \frac{\partial f_{ij}}{\partial k_m} d\theta \right).$$

The tensor K_{ij} can be expressed in terms of second-order orientation moments. To do this we use the identity [16] $U_{lm} \int k_l \frac{\partial F_{ij}}{\partial k_m} d\mathbf{k} = 0$, which can be transformed with the help of (5) to

$$K_{ij} = -3U_{lm} f_{ij}^{(lm)}. \tag{14}$$

The relation (13) is an energy balance equation for the largest ($k \ll r_c^{-1}$) vortices. Mathematically, it is a differential equation for the tensor $f_{ij}^{(0)}$ which, however, also involves the second-order moments.

Using (13) and (14), we can rewrite (12) in the simpler form

$$\left(U_k \frac{\partial \ln r_c}{\partial x_k} f_{ij}^{(0)} - U_{lm} f_{ij}^{(lm)} \right) k \frac{d\varphi}{dk} + (f_{il,j}^{(l)} - f_{lm,j}^{(lmi)})_s kt + 2\eta k^2 f_{ij}^{(0)} \varphi = 0. \tag{15}$$

In the special case when $U_{\ell m} = 0$ and the turbulence is isotropic, i.e. $f_{ij}^{(0)} \sim \delta_{ij}$ and $f^{(\ell)}_{i\ell,j} \sim \delta_{ij}$, using (1) and (2) we can easily reduce (15) to the equation

$$\tau \bar{k}\, d\varphi/d\bar{k} + \bar{r}_c^a \bar{k} t + \bar{k}^2 \varphi = 0, \tag{16}$$

which is essentially the Carman–Hovart equation [33] written in spectral form with the use of the scaling hypothesis (1).

Writing (15) in dimensionless form and comparing with (16), we obtain

$$t_d \left(U_k \frac{\partial \ln r_c}{\partial x_k} f_{ij}^{(0)} - U_{lm} f_{ij}^{(lm)} \right) = 2\tau' f_{ij}^{(0)} \equiv 2\bar{r}_c^{-1/\nu} f_{ij}^{(0)}; \tag{17}$$

$$(\bar{f}_{il,j}^{(l)} - \bar{f}_{lm,j}^{(lmi)})_s = -2\bar{r}_c^{a} \bar{f}_{ij}^{(0)} \tag{18}$$

(τ' is the "temperature" of the shear flow). It follows from (17) that in the absence of shear τ' is equal to τ. When $U_{ij} = 0$, (17) reduces to an algebraic relation between the zero-order and second-order moments of f_{ij}.

Integrating (15) over all $\mathbf{k}$, we obtain the turbulent energy balance equation. Using (17) and assuming that the fluctuations are isotropic in the dissipative region, it can be reduced to the form

$$2\langle\varepsilon\rangle \delta_{ij}/3 - \Phi_{ij,1} = 3\alpha r_c^{-3} \bar{r}_c^{-1/\nu} t_d^{-1} f_{ij}^{(0)}. \tag{19}$$

Contracting indices with the help of (7), we obtain the algebraic relation

$$\langle\varepsilon\rangle = 3 t_d^{-1} \bar{r}_c^{-1/\nu} \langle u_i^2 \rangle / 2. \tag{20}$$

Finally, using (20) it is not difficult to transform (19) to

$$\Phi_{ij,1} = -2 \frac{\langle\varepsilon\rangle}{\langle u_k^2 \rangle} \left(\langle u_i u_j \rangle - \frac{1}{3} \langle u_k^2 \rangle \delta_{ij} \right). \tag{21}$$

This last relation is equivalent to the Rott formula if we set the Rott constant C equal to 2 and choose the quantity $\langle u_k^2 \rangle / \langle\varepsilon\rangle$ as the scale of time; C = 2 means that nonlinear interactions do not participate in the process of return to isotropy [5, 10]. The latter is consistent with the quasilinear form of (13).

Summarizing, using the assumptions (3) and (5), (6), we can reduce the equation for the tensor $\langle u_i u_j \rangle$ (or equivalently $f_{ij}^{(0)}$, to within the factor r_c^{-3}) to the much simpler form (13) and the algebraic relations (17), (20), and (21). This does not completely solve the problem of closure and the selection of the secular quantities, however, since (13) involves different linear combinations of the components of the tensor $f_{ij}^{(\ell m)}$: longitudinal ($U_{\ell m}f_{ij}^{(\ell m)}$) and transverse ($P_{ij} \equiv U_{\ell m}f_{mj}^{(\ell i)}$) products with $U_{\ell m}$. The longitudinal quantities can be expressed directly in terms of the zero-order moments using (17), but in general this formula is insufficient to determine all of the components of P_{ij}.

In this case it is natural to widen the set of secular quantities and include the second-order moments $f_{ij}^{(pq)}$, in addition to r_c, $\langle\varepsilon\rangle$, $f_{ij}^{(0)}$. Equations for the second-order moments are obtained from (8) after multiplication by $\theta_p\theta_q$ and integration over all θ:

$$U_k\frac{\partial f_{ij}^{(pq)}}{\partial x_k}\varphi+U_k\frac{\partial\ln r_c}{\partial x_k}f_{ij}^{(pq)}k\frac{d\varphi}{dk}+\left(U_{il}f_{lj}^{(pq)}\right)_s\varphi+K_{ij}^{(pq)}-U_{lm}f_{ij}^{(lmpq)}k\frac{d\varphi}{dk} \tag{22}$$

$$+\left(f_{il,j}^{(lpq)}-f_{lm,j}^{(lmipq)}\right)_s kt+2\eta k^2 f_{ij}^{(pq)}\varphi=2U_{lm}\left(f_{mj}^{(lipq)}\right)_s\varphi\quad\left(K_{ij}^{(pq)}=-U_{lm}\int k_l\theta_p\theta_q\frac{\partial f_{ij}}{\partial k_m}d\theta\right).$$

An expression for $K_{ij}^{(pq)}$ can be obtained in the same way as was done for K_{ij}: integrating by parts and using (10), the quantity $J_{ij}^{(pq)}=U_{lm}\int\theta_p\theta_q k_l\frac{\partial F_{ij}}{\partial k_m}d\mathbf{k}$ can be represented in the form

$$J_{ij}^{(pq)}=-\alpha r_c^{-3}\left(U_{lp}f_{ij}^{(lq)}+U_{lq}f_{ij}^{(lp)}-2U_{lm}f_{ij}^{(lmpq)}\right),$$

but using (15) we find

$$J_{ij}^{(pq)}=\alpha r_c^{-3}\left(-K_{ij}^{(pq)}-3U_{lm}f_{ij}^{(lmpq)}\right).$$

Equating these last two expressions, we obtain

$$K_{ij}^{(pq)}=-5U_{lm}f_{ij}^{(lmpq)}+\left(U_{lq}f_{ij}^{(lp)}\right)_s. \tag{23}$$

Further analysis of (22) is analogous to (12). It follows from (22) that in the limit $kr_c \to 0$

$$U_k\frac{\partial f_{ij}^{(pq)}}{\partial x_k}+\left(U_{il}f_{lj}^{(pq)}\right)_s+K_{ij}^{(pq)}=2U_{lm}\left(f_{mj}^{(lipq)}\right)_s. \tag{24}$$

Using (24), (22) takes the form

$$\left(U_k\frac{\partial\ln r_c}{\partial x_k}f_{ij}^{(pq)}-U_{lm}f_{ij}^{(lmpq)}\right)k\frac{d\varphi}{dk}+\left(f_{il,j}^{(lpq)}-f_{lm,j}^{(lmipq)}\right)_s kt+2\eta k^2 f_{ij}^{(pq)}\varphi=0. \tag{25}$$

Comparing with (16), we have

$$t_d\left(U_k\frac{\partial\ln r_c}{\partial x_k}f_{ij}^{(pq)}-U_{lm}f_{ij}^{(lmpq)}\right)=2\bar{r}_c^{-1/\nu}f_{ij}^{(pq)}; \tag{26}$$

$$\left(\bar{f}_{il,j}^{(lpq)}-\bar{f}_{lm,j}^{(lmipq)}\right)_s=-2\bar{r}_c^{-a}\bar{f}_{ij}^{(pq)}. \tag{27}$$

Like (18), the result (27) is an algebraic relation between the moments of the functions f_{ij} and $f_{i\ell,j}$. The relation (26) has the same structure as (17) and illustrates the general rule that contraction of two upper indices of an orientation tensor $f_{ij}^{(\ell\ldots m)}$ with $U_{\ell m}$ lowers its order by two.

With the help of (26) we can obtain information on the components of the tensor P_{ij} sufficient for closure. Indeed, using (23) and (26), (24) can be rewritten in the form

$$U_k\frac{\partial f_{ij}^{(pq)}}{\partial x_k}+\left(U_{il}f_{lj}^{(pq)}\right)_s+\left(U_{lq}f_{ij}^{(lp)}\right)_s-5Af_{ij}^{(pq)}=2U_{lm}\left(f_{mj}^{(lipq)}\right)_s. \tag{28}$$

Contracting in (28) the indices p, i with U_{pi} and q, j with U_{qj}, and using (26), we obtain

$$U_k\frac{\partial(\widehat{P}\widehat{U})}{\partial x_k}+4\widehat{P}\widehat{U}^2-9A\widehat{P}\widehat{U}=2U_k f_{ij}^{(pq)}\left(U_{qj}\frac{\partial U_{pi}}{\partial x_k}+U_{pi}\frac{\partial U_{qj}}{\partial x_k}\right). \tag{29}$$

Here

$$A=U_k\frac{\partial\ln r_c}{\partial x_k}-2\bar{r}_c^{-1/\nu}t_d^{-1}, \tag{30}$$

and the notation $\hat{P}\hat{U}$ indicates the tensor product.

The system of equations (13), (17), (20), (29) for $\langle\varepsilon\rangle$, r_c, $f_{ij}^{(0)}$, P_{ij} (or equivalently, r_c, $\langle\varepsilon\rangle$, $\langle u_i u_j\rangle$, $\Phi_{ij,2}$, according to (7) and (9)) is fundamental for classes of turbulent flow considered here. The system contains only the single constant α, which can be determined independently from spectral data.

We note that closure was obtained using only the zero and second-order moments. Higher-order moments can be expressed in terms of the lower-order moments algebraically. But the differential equation for $\hat{P}\hat{U}$ shows that the attempts of [5, 24-26, 34] to give a local description of turbulence using only the fields $\langle\varepsilon\rangle$, r_c, $\langle u_i u_j\rangle$ are not completely correct.

To test the method we apply the above system of equations to accelerating flow behind a grid in an axisymmetric converging tube. This type of channel is widely used to improve turbulent characteristics in aerodynamic tubes [19, 20, 35-38], however the experimental data are widely scattered and are sometimes inconsistent with one another [20, 38]. There are also unresolved theoretical questions in this case. For example, it remains essentially unknown whether the intensity $\langle u_1^2\rangle$ of longitudinal turbulent fluctuations increases or decreases as a result of contraction [39].

The average velocity field in this problem can be considered as given: it is determined only by the geometry of the channel walls. For simplicity we consider the turbulent characteristics along the channel axis. Let x and U be the longitudinal coordinate and the corresponding component of the average velocity. Then the average deformation rate tensor has the simple form

$$U_{ij} = 3\varkappa\delta_{i1}\delta_{j1} - \varkappa\delta_{ij}, \quad \varkappa(x) = \frac{1}{2}\frac{\partial U}{\partial x},$$

the tensors $f_{ij}^{(0)}$ and P_{ij} are diagonal and the partial differential equations reduce to ordinary differential equations.

It can be shown that the products $\hat{P}\hat{U}$ and $\hat{P}\hat{U}^2$ can be expressed in terms of the single independent component P_{11} of the tensor P_{ij}:

$$\widehat{P}\widehat{U} = 3\varkappa P_{11}, \quad \widehat{P}\widehat{U}^2 = 3\varkappa^2 P_{11}. \tag{31}$$

Using the identity $f_{\ell j}^{(\ell i)} = 0$, which follows from the incompressibility condition, the quantity P_{11} can be written in the form

$$P_{11} = 3\varkappa f_{11}^{(11)}. \tag{32}$$

On the other hand, using the relation $f_{ij}^{(\ell\ell)} = f_{ij}^{(0)}$ and the definition (30), we obtain from (17)

$$3\varkappa f_{ij}^{(11)} = (A + \varkappa) f_{ij}^{(0)}. \tag{33}$$

Comparing (33) with i = j = 1 and (32), we have

$$P_{11} = (A + \varkappa) f_{11}^{(0)}. \tag{34}$$

Using (31), (32), and (34), the system of equations for the secular quantities takes the form

$$U\frac{df_{11}^{(0)}}{dx} = 7Af_{11}^{(0)}, \quad U\frac{df_{ii}^{(0)}}{dx} = (3A + 2\varkappa) f_{ii}^{(0)} - 6\varkappa f_{11}^{(0)}, \tag{35}$$

$$U\frac{d}{dx}\left((A + \varkappa)\varkappa f_{11}^{(0)}\right) + 4\varkappa^2 (A + \varkappa) f_{11}^{(0)} - 9A\varkappa (A + \varkappa) f_{11}^{(0)} = 2U\frac{d\varkappa}{dx}(A + \varkappa) f_{11}^{(0)},$$

$$U\frac{dr_c}{dx} = Ar_c + 2r_c\bar{r}_c^{-1/\nu} t_d^{-1}.$$

Using the algebraic representation (20) for $\langle\varepsilon\rangle$, it is not difficult to show that this system is closed for the functions r_c, $f_{11}^{(0)}$, $f_{ii}^{(0)}$, and A. We next consider its solution.

From the first and third equations we obtain the "integral of the motion"

$$\frac{|A + \varkappa| U^2}{\varkappa \left(f_{11}^{(0)}\right)^{2/7}} = b = \text{const}, \tag{36}$$

from which we can eliminate the variable A from the system of equations. After some straightforward calculations and changes of variables, we obtain a Bernoulli equation for $f_{11}^{(0)}$

$$\frac{df_{11}^{(0)}}{dU} + \frac{7}{2}\frac{f_{11}^{(0)}}{U} - \frac{7}{2}\frac{b}{U^3}\left(f_{11}^{(0)}\right)^{9/7} = 0. \tag{37}$$

Its solution is written as

$$f_{11}^{(0)} = \left(\frac{1+\beta}{1+\beta c^3}\right)^{7/2} c^7, \tag{38}$$

where $\beta = 3U^2(0)/b - 1$ is a constant; $c = U(x)/U(0)$ is the effective convergence of the flow; $x = 0$ corresponds to the initial cross section of the converging tube. Here and below we present solutions normalized to the initial value. For example, $f_{11}^{(0)} = 1$ for $c = 1$.

Substituting (38) for $f_{11}^{(0)}$ into (36), we have

$$A = 3\varkappa/(1 + \beta c^3) - \varkappa. \tag{39}$$

Finally, the two remaining equations of the system (35) are easily solved with the help of (38) and (39). We obtain the following results for the parameters of greatest practical interest: the intensity of longitudinal fluctuations, the correlation length, and the anisotropy parameter $\langle u_i^2\rangle/\langle u_1^2\rangle$:

$$\langle u_1^2\rangle = c^4\left(\frac{1+\beta}{1+\beta c^3}\right)^2\left[1 + \frac{B}{L(1+\beta)^{1/2}}\int_0^x \left((\gamma\beta - \beta - 1) + (\gamma + \beta + 1)\,c^{-3}\right)^{1/2} dx\right]^{-6/5}; \tag{40}$$

$$r_c = (\langle u_1^2\rangle)^{-1/3} c^{7/3}\left(\frac{1+\beta}{1+\beta c^3}\right)^{7/6}; \tag{41}$$

$$\langle u_i^2\rangle/\langle u_1^2\rangle = \frac{1+\beta c^3}{c^3(1+\beta)^2}\left((\gamma\beta - \beta - 1)\,c^3 + (\gamma + \beta + 1)\right). \tag{42}$$

Here L is the length of the contraction part of the flow; $B = 5\sqrt{\langle u_1^2(0)\rangle}L/(U(0)r_c(0))$ is the ratio of the "external" time scale L/U to the time scale r_c/u of energy-containing turbulent fluctuations; $\gamma = \langle u_i^2(0)\rangle/\langle u_1^2(0)\rangle$. For isotropic initial conditions $\gamma = 3$.

We analyze the results of the calculation and formulate some conclusions.

1. When $c = 1$ (no contraction) (40) and (41) reduce to the earlier-obtained power-law relations describing the decay of turbulence behind a grid.

2. Substituting (39) for A into (34), we obtain $P_{11} = \frac{3\varkappa}{1+\beta c^3} f_{11}^{(0)}$ or equivalently

$$\Phi_{11,2} = 3\langle u_1^2\rangle \frac{dU}{dx_1}\Big/(1 + \beta c^3).$$

This last formula illustrates the difference between our method and the models cited above, in which (9) is postulated for $\Phi_{ij,2}$.

3. The relative simplicity with which we were able to integrate the system (35) results from the quasilinear (in the orientation moments) nature of the original equations. In this connection it is natural to compare our method of calculation to the fast distortion theory.

In the limit $B \ll 1$, which corresponds to the assumption of "fast" distortion, we have from (40) and (42) $\langle u_1^2(x)\rangle \sim c^{-2}$, $\langle u_i^2(x)\rangle \sim c$ for large c. These results are consistent with the results from the fast distortion theory and with the earlier estimates of Prandtl and Taylor [20]. However this agreement should not be overemphasized, since the formal limit $B \ll 1$ is not realizable physically: experimentally one obtains in the best case values of B close to unity.

The difference between the two methods is much more important. In the fast distortion theory the nonlinear and dissipative terms are simply eliminated from the spectral equations. In our method these terms are taken into account and play a fundamental role in the determination of the universal spectral functions, whose properties in the long-wavelength region (scaling, factorization, and the limiting behavior when $k \to 0$) make it possibe to close the system of equations for the basic turbulent characteristics and determine its quasilinear nature. It is not actually necessary to use the explicit form of the k-dependence of the spectral functions, in contrast to the fast distortion theory, where the explicit forms are required to formulate the initial conditions.

4. The solution of the system (35) includes the parameter β. According to (36) and the relation $U_{\ell m} f_{ij}^{(\ell m)} = A f_{ij}^{(0)}$, this parameter is determined by the initial values of the zeroth and second-order orientation moments. The direct measurement of $f_{ij}^{(\ell m)}$ is a

complicated problem, however. In addition, there is the fundamental difficulty that if the turbulence on entry is isotropic in the sense that $F_{ij} \sim p_{ij}$, then the condition $U_{\ell m} f_{ij}(\ell m) \sim f_{ij}(0)$ obviously cannot be satisfied. The resolution of this contradiction is that an isotropic parameterization cannot be used in general for the tensor $f_{ij}^{(\ell m)}$, even if $f_{ij}^{(0)}$ is isotropic. This is because the average shear affects all orientation moments independently (this has been shown here for the zero and second-order moments). Turbulence produced by the shear when the fluid passes through the grid cannot be eliminated and therefore the problem is not as simple as normally assumed: even if $\langle u_i u_j \rangle \sim f_{ij}^{(0)} \sim \delta_{ij}$, the isotropy condition $F_{ij} \sim p_{ij}$ does not automatically follow.

In this sense it is appropriate to speak of turbulence having memory and the memory can be retained even in the higher-order moments in the form of nonlocal dependence on U_{ij}. Isotropic turbulence must therefore be considered only as a very crude model of real turbulent flow, which is never realized in a literal sense. One concludes for flows in converging tubes that the value of the parameter β and the quantitative results of contraction depend essentially on the type of grid and the level of turbulence in the initial cross section of the nozzle. This conclusion is supported directly by experiment and explains the large scatter in the experimental data.

5. For direct comparison with experiment it is simplest to find β from (42). For example, we obtain a value of β slightly greater than 0.5 from the results of [20] with γ = 3. In this case the basic turbulent characteristics depend monotonically on x.

More complex behavior has been noted in a number of experimental papers. For example, $\langle u_1^2 \rangle$ was observed to decrease in [19, 20] only up to the cross section with c = 4, after which the intensity of longitudinal fluctuations grew. However, as noted in [38], this effect can occur because of acoustic perturbations on the outlet from the converging tube.

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